Eur. Phys. J. D **37**, 241–245 (2006)
DOI: 10.1140/epjd/e2005-00277-3

THE EUROPEAN PHYSICAL JOURNAL D

The dynamics of ring dark soliton in inhomogeneous Bose-Einstein condensates

J.-K. Xue[a]

Physics and Electronics Engineering College, Northwest Normal University, Lanzhou 730070, P.R. China

Received 18 March 2005 / Received in final form 21 May 2005
Published online 18 October 2005 –

Abstract. The dynamics of the ring dark soliton in an inhomogeneous Bose-Einstein condensates (BEC) with thin disk-shaped potential trapping is investigated analytically and numerically. Analytical result shows that the ring dark soliton is governed by a variable coefficients Korteweg-de Vries (KdV) equation. The effect of the ring curvature (nonplanar geometry) and the inhomogeneous of the background on soliton amplitude and the emitted radiation profiles are obtained analytically. The theoretical results are confirmed by the direct numerical results.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 03.75.Kk Dynamic properties of condensates; collective and hydrodynamic excitations, superfluid flow

Solitons are fundamental non-linear coherent structures in Bose-Einstein condensates (BEC). Experimental and theoretical investigations for nonlinear evolution of solitons in BEC have been paid increasing interests. Both bright [1] and dark solitons [2] are observed and investigated in BEC. Recently [3], the concept of ring dark soliton in BEC is introduced and the ring dark soliton in a BEC with disk-shaped trap is studied, and predicted the existence of both oscillatory and stationary ring dark solitons.

Soliton's dynamics, stability, and dissipation are important issues of experimental and theoretical consideration in BEC. Because of the interaction with the thermal cloud, soliton experiences thermal instabilities [4] and this causes energy dissipation from soliton. The experimentally observed dynamical instability [5] of dark soliton is due to their quesi-1D character: when embedded in a higher dimension, a dark soliton stripe becomes unstable against transverse perturbation and will bend via the snaking instability and ultimately decay into vortices. However, the instability band of the dark stripe solitons, characterized by a maximum perturbation wavenumber $Q_{\max}$, may be suppressed by bending a dark stripe to close it into an annulus of length $L < 2\pi/Q_{\max}$ [6]. On the other hand, the soliton is expected to have a circular symmetry and the stable ring dark soliton can be observed. Another important dynamical instability is caused by the inhomogeneity of the system. The longitudinal confinement featured in BEC experiments results in the instability when the soliton moves through the inhomogeneous background density, and leads to decay of soliton via sound emission from the soliton [7–10]. Certainly, the ring dark soliton in trapped BEC is also experiences the instability of sound emission. We also can expect that the ring dark soliton also experiences the instability arises from the ring curvature (i.e., the non-planar geometry effect introducing another inhomogeneity). Hence, it is important to study the dynamics and instability of ring dark soliton in BEC. However, the dynamics, the instability to sound emission and the profiles of the emitted sound waves of the ring dark soliton remains an open question in BEC. It is the aim of the present paper to discuss these points analytically in a BEC with disk-shaped trap. By using a perturbation method, we show that the ring dark soliton in the radial symmetric cylindrical BEC is governed by a variable coefficients Korteweg-de Vries (KdV) equation. The reduction to the KdV like equation may be useful to understand the dynamics of ring dark soliton in trapped BEC. The analytical expressions for the evolution of soliton, emitted radiation profiles, and soliton oscillation velocity are obtained.

The evolution of the weakly coupled BEC at low temperature is governed by the time-dependent Gross-Pitaevskii (GP) equation with the external disk-shaped potential $V(r)$

$$i\frac{\partial \Psi}{\partial t} = \left[-\frac{1}{2}\nabla^2 - \frac{1}{2}\frac{\partial^2}{\partial z^2} + V(r) + Q|\Psi|^2\right]\Psi, \qquad (1)$$

and the disk-shaped trap has the form

$$\begin{aligned} V(r,z) &= V(r) + V(z) \\ &= \frac{1}{2}\left(n_0^{1/3} a_r\right)^{-4} r^2 + \frac{1}{2}\left(n_0^{1/3} a_z\right)^{-4} z^2, \end{aligned} \qquad (2)$$

[a] e-mail: Xuejk@nwnu.edu.cn

where $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r$ and $\int dV|\Psi|^2 = N$, with N the number of atoms in BEC and $n_0 = |\Psi_0|^2$ the peak density of the gas. $a_r = [\hbar/(m\omega_r)]^{1/2}$ and $a_z = [\hbar/(m\omega_z)]^{1/2}$ are the harmonic oscillator length in the axial and transverse directions. ω_r and ω_z are frequencies of the trap in the radial r and z direction, respectively. $Q = 4\pi a_s n_0^{1/3}$, with a_s the s-wave scattering length, m the mass of the atom. Wave function Ψ, time t, and variables (r, z) are normalized by $\sqrt{n_0}$, $m/(\hbar n_0^{2/3})$, and $n_0^{-1/3}$, respectively.

In order to investigate the dynamics of the ring solitary wave in BEC, we consider the axially symmetric solitary waves traveling along the radial direction in a BEC with disk-shaped trap. The excitation is created in the BEC with very thin disk-shaped trap, i.e., the case in which the trapping potential in r-direction is much weaker than that in z-direction, mathematically, $\omega_r/\omega_z \ll 1$, i.e., $a_z \ll a_r$. This means that the motion of atoms in the z-direction is essentially frozen and is governed by the ground-state wave function of the corresponding harmonic oscillator. Hence, the excitations can propagate only in r-direction. According to the above assumption, we can set [11]

$$\Psi(r, z, t) = G_0(z)\Phi(r, t), \tag{3}$$

where $G_0(z) = \exp[-(n_0^{1/3}a_z)^{-2}z^2/2]$ is the ground-state wave function of the 1D harmonic oscillator with the potential $(n_0^{1/3}a_z)^{-4}z^2/2$ in the z-direction. Then, substituting equation (3) into equation (1), we obtain

$$i\frac{\partial\Phi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(r) + Q^{'}|\Phi|^2\right]\Phi, \tag{4}$$

where $\int|\Phi|^2dxdy = N/(\sqrt{\pi}n_0^{1/3}a_z)$ and $Q' = I_0 Q$ is an effective interaction constant with $I_0 = \int_{-\infty}^{\infty}dzG_0^4(z)/\int_{-\infty}^{\infty}dzG_0^2(z) = 1/\sqrt{2}$. Because the contribution of the higher-order eigenmodes of the harmonic oscillator in the z-direction is very small and can be safely neglected, so, in deducing equation (4), we have multiplied equation (1) by $G_0^*(z)$ and then integrated once with respect to z to eliminate the dependence on z [11,12]. Now we seek for solution to equation (4) in the form

$$\Phi(r, t) = A(r, t)\exp[-i\mu t + i\phi(r, t)], \tag{5}$$

where μ is the chemical potential of the condensate and ϕ is a phase function contributed from the excitation. Then, equation (4) reduces to

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial r}\frac{\partial\phi}{\partial r} + \frac{A}{2}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) = 0, \tag{6}$$

$$-\frac{1}{2}\left(\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right) - \mu A + \left[\frac{\partial\phi}{\partial t} + V(r) + \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2\right]A + Q'A^3 = 0. \tag{7}$$

We now employ the standard reductive perturbation method [13,14] to obtain the nonlinear evolution of the ring dark soliton in BEC with weak nonlinear interaction. The reductive perturbation method is successfully used to study the nonlinear excitation in fluid physics, plasma physics, and nonlinear optics [13–16]. In particular, the validity and usefulness of this perturbation method is also demonstrated and confirmed in studying the dynamics of shallow quasi-one dimensional solitons in BEC [8,10,11]. According to the reductive perturbation method, in order to obtain a self-consistent asymptotic expansion in disk-shaped BEC, we define a small parameter $\epsilon = \Omega^{-2/3}(n_0^{1/3}a_r)^{-4/3}$ (because $(n_0^{1/3}a_r)^{-4}$ is small, Ω is a parameter and, as see below, expresses the normalized strength of the axial trap frequency) and two slow variables $R = \epsilon^{3/2}r$, $\xi = \epsilon^{1/2}[t - \int C(R)dr]$, where $C(R)$ is the inverse of the local velocity of the nonlinear excitations. Then, the external potential $V(r)$ takes the form

$$V(R) = \frac{1}{2}\Omega^2R^2. \tag{8}$$

For weak non-linear excitations, the dependent variables can be scaled as

$$A = A_0(R) + \epsilon A_1(R, \xi) + \epsilon^2 A_2(R, \xi) + \cdots, \tag{9}$$

$$\phi = \epsilon^{1/2}\phi_1(R, \xi) + \epsilon^{3/2}\phi_2(R, \xi) + \cdots, \tag{10}$$

when $\epsilon \to 0$, equations (9) and (10) indicate that the condensate background $A_0(R)$, i.e., without perturbation, is recovered.

Then, substituting the above expansions into equations (6, 7) and collecting the terms in the different powers of ϵ, we can obtain each nth-order reduced equation. To the leading order, $O(1)$, we obtain the background expression

$$A_0(R) = \sqrt{[\mu - V(R)]/Q'}, \tag{11}$$

To the next order, we have

$$\frac{\partial A_1}{\partial\xi} + \frac{1}{2}C^2A_0\frac{\partial^2\phi_1}{\partial\xi^2} = 0, \tag{12}$$

$$2Q'A_0A_1 + \frac{\partial\phi_1}{\partial\xi} = 0, \tag{13}$$

which results in

$$A_1 = -\frac{1}{2Q'A_0}\frac{\partial\phi_1}{\partial\xi}, \tag{14}$$

and $C^2 = 1/Q'A_0^2$ is also obtained by plugging equation (14) into equation (12). The expression of $C^{-1} = \sqrt{Q'}A_0$ actually is the same as the speed of sound, i.e., $C^{-1} \equiv c_s = \sqrt{Q'}A_0$, which can also be easily obtained from the dispersion relation of the excitations described by equations (6, 7) [11]. At the next order, we have

$$\frac{\partial A_2}{\partial\xi} + \frac{1}{2}C^2A_0\frac{\partial^2\phi_2}{\partial\xi^2} + \frac{1}{2}C^2A_1\frac{\partial^2\phi_1}{\partial\xi^2} - CA_0\frac{\partial^2\phi_1}{\partial\xi\partial R} - C\left(\frac{\partial A_0}{\partial R} + \frac{A_0}{2R} - C\frac{\partial A_1}{\partial\xi}\right)\frac{\partial\phi_1}{\partial\xi} = 0, \tag{15}$$

$$2Q'A_0A_2 + \frac{\partial \phi_2}{\partial \xi} + \frac{1}{2}C^2\left(\frac{\partial \phi_1}{\partial \xi}\right)^2 - \frac{C^2}{2A_0}\frac{\partial^2 A_1}{\partial \xi^2} + Q'A_1^2 = 0. \tag{16}$$

By using equation (14) and eliminating A_2 and ϕ_2 from equations (15) and (16), we obtain

$$\frac{\partial A_1}{\partial R} - \alpha A_1\frac{\partial A_1}{\partial \xi} + \beta\frac{\partial^3 A_1}{\partial \xi^3} + \left(\frac{1}{2R} + \frac{d}{dR}\ln A_0\right)A_1 = 0, \tag{17}$$

where $\alpha = 3C/A_0, \beta = C^5/8$. Equation (17) is a variable coefficients KdV equation describing the small-amplitude ring dark soliton in the radial inhomogeneous BEC. The last term in equation (17) refers to the combined effect of ring curvature and inhomogeneous caused by the trapping. In the absence of the trapping, equation (17) reduces to the ordinary cylindrical KdV equation describing the small-amplitude ring dark soliton in the radial homogeneous BEC. It is important to note that, because there are terms like R^{-1} and $A_0^{-1}(R)$ in the above derived expressions, our perturbation results (including Eq. (17)) are suitable for the cases far away from the cloud boundaries and the center. Indeed, the present paper is focused on considering the evolution of ring dark soliton under this condition.

Introducing the transformations $A_1 = (6\beta/\alpha)u$, $\chi = \int \beta dR$, equation (17) reduces to

$$\frac{\partial u}{\partial \chi} - 6u\frac{\partial u}{\partial \xi} + \frac{\partial^3 u}{\partial \xi^3} = P(\chi)u, \tag{18}$$

where

$$P(\chi) = -\frac{d}{d\chi}\ln\left(\frac{\beta}{\alpha}\right) - \frac{1}{\beta}\left(\frac{1}{2R} + \frac{d}{dR}\ln A_0\right).$$

For a planar ($R \to \infty$, $R^{-1} \to 0$) and homogeneous BEC ($d/dR = d/d\chi = 0$ in $P(\chi)$), i.e., $P(\chi) = 0$, equation (18) is a one-dimensional KdV equation, which has a single-soliton solution of the form

$$u = -2\kappa^2 \mathrm{sech}^2 Z, \quad Z = \kappa[\xi - \vartheta(\chi)], \tag{19}$$

where $\vartheta(\chi) = 4\kappa^2\chi + \vartheta_0$ is the soliton center (with $d\vartheta/d\chi = 4\kappa^2$ being the soliton velocity in the $\xi - \chi$ reference frame), while κ and ϑ_0 are arbitrary constants presenting the soliton's amplitude and initial position, respectively.

Now we discuss the evolution of ring dark soliton of equation (18) in inhomogeneous BEC, i.e., in the presence of the external trapping potential. An analytical solution of equation (18) can be obtained by using a suitable perturbation theory for soliton. In regions of small density gradients (this can be satisfied by the local density approximation) and for larger ring radius (this is the most interesting case), the right hand side of equation (18) can be treated as a small perturbation. The solution of equation (18) can be expressed as [17,18]

$$u = u_s + u_r \tag{20}$$

where u_s is the soliton part, which has the same functional form as in the unperturbed homogeneous case (cf., Eq. (19)), but with the soliton parameters κ and ϑ being now unknown functions of χ. The contribution u_r, being of the same order of the smallness as $P(\chi)$, denotes the radiation part of the solution due to the effect of axial inhomogeneity. According to [17,18], the soliton's amplitude $\kappa(\chi)$ and center $\vartheta(\chi)$ are determined by

$$\frac{d\kappa}{d\chi} = \frac{2}{3}\kappa P(\chi), \quad \frac{d\vartheta}{d\chi} = 4\kappa^2 + \frac{1}{3\kappa}P(\chi) \tag{21}$$

Integrating of equation (21) and with the help of equations (8) and (11), one can obtain the soliton's amplitude $\kappa(\chi)$ and center $\vartheta(\chi)$ expressed in terms of the slow variable R

$$\kappa(R) = \kappa(R_0)\left(\frac{R_0}{R}\right)^{1/3}\left[\frac{A_0(R)}{A_0(R_0)}\right]^{4/3}, \tag{22}$$

$$\begin{aligned}\vartheta(R) = &-\frac{1}{6\kappa(R_0)}R_0^{-1/3}A_0^{4/3}(R_0)\int R^{-2/3}A_0^{-4/3}(R)dR \\ &+ \frac{\kappa^2(R_0)}{4Q'^{5/2}}R_0^{2/3}A_0^{-8/3}(R_0)\int R^{-2/3}A_0^{-7/3}(R)dR \\ &- \frac{\Omega^2}{3\kappa(R_0)Q'}R_0^{-1/3}A_0^{4/3}(R_0)\int R^{4/3}A_0^{-10/3}(R)dR\end{aligned} \tag{23}$$

where $\kappa(R_0)$ and $A_0(R_0)$ are the soliton amplitude and density at $R = R_0$, respectively. It is clear that, because of the ring curvature (nonplanar geometry) and the inhomogeneous of the background, the ring dark soliton amplitude vary according to $(R_0/R)^{1/3}[A_0(R)/A_0(R_0)]^{4/3}$ law as it propagates. Following [17], the radiation part of the soliton is expressed by

$$u_r = -\frac{8Q'^{5/2}A_0^5(R)}{3\kappa(R)}\frac{d}{dR}\{\ln[R^{1/2}A_0^{-2}(R)]\}F(Z) \tag{24}$$

$$\begin{aligned}F(Z) = &\left[1 + Z - \left(\frac{Z^2}{2} + Z + \frac{\pi^2}{24} + \frac{3}{4}\right)\tanh Z\right]\mathrm{sech}^2(Z) \\ &- \frac{1}{2}(1 - \tanh Z).\end{aligned} \tag{25}$$

One can find from equation (24) clearly that the sound emission from soliton is caused by the ring curvature (i.e., nonplanar geometry introducing another inhomogeneity) and the inhomogeneous of the background. When the ring soliton propagates in inhomogeneous BEC, a continual energy transfer from the soliton to this radiation wave occurs. On the other hand, energy carried by this radiation wave expresses the energy emitted by the soliton [18]. If the background is homogeneous (i.e., $d/dR = 0$), then the radiation will vanishes.

Taking into account the transformation $A_1 = (6\beta/\alpha)u$ and equations (9), (20) and (14), we have a first order solution of equations (6, 7) as

$$A = A_0 + \epsilon A_1$$
$$= A_0 \left[1 - \frac{1}{2}\frac{\epsilon\kappa^2(R)}{Q'^2 A_0^4(R)}\text{sech}^2(Z)\right] - \frac{2\epsilon Q'^{1/2}A_0^2(R)}{3\kappa(R)}$$
$$\times \frac{d}{dR}\left\{\ln\left[R^{1/2}A_0^{-2}(R)\right]\right\}F(Z) \quad (26)$$
$$\phi = \frac{\epsilon^{1/2}\kappa(R)}{Q' A_0^2(R)}\tanh(Z) + \frac{4\epsilon^{1/2}Q'^{3/2}A_0^3(R)}{3\kappa(R)}$$
$$\times \frac{d}{dR}\{\ln[R^{1/2}A_0^{-2}(R)]\}\int d\xi F(Z). \quad (27)$$

If the radiation loss is neglected, then equations (26, 27) reduce to

$$A = A_0\left[1 - \frac{1}{2}\frac{\epsilon\kappa^2(R)}{Q'^2 A_0^4(R)}\text{sech}^2(Z)\right] \quad (28)$$
$$\phi = \frac{\epsilon^{1/2}\kappa(R)}{Q' A_0^2(R)}\tanh(Z). \quad (29)$$

Then, the GP equation (4) has the solution of the form

$$\Phi = A_0(R)\left[1 - \frac{1}{2}\frac{\epsilon\kappa^2(R)}{Q'^2 A_0^4(R)}\text{sech}^2(Z)\right]\exp(-i\mu t + i\phi). \quad (30)$$

Now we discuss the variation law of the amplitude of the excitation (soliton), i.e., $|A - A_0|$, under the effect of sound emission. According to equation (26), the peak value of $|A - A_0|$ occurs near $Z = 0$, so, we expand equation (26) about $Z = 0$ to $O(Z^2)$ with the result

$$|A - A_0| = \frac{1}{2}\frac{\epsilon\kappa^2(R)}{Q'^2 A_0^3(R)}(1 - Z^2)$$
$$+ \frac{\epsilon Q'^{1/2}A_0^2(R)}{3\kappa(R)}\frac{d}{dR}\{\ln[R^{1/2}A_0^{-2}(R)]\}$$
$$\times\left[1 + \frac{1}{2}\left(3 - \frac{\pi^2}{6}\right)Z + 4Z^2\right]. \quad (31)$$

From $d|A - A_0|/dZ = 0$, we find that the peak occurs at $Z = Z_m$ with

$$Z_m = \frac{\epsilon Q'^{1/2}A_0^2(R)}{6\kappa(R)}\left(3 - \frac{\pi^2}{6}\right)$$
$$\times \frac{d}{dR}\left\{\ln[R^{1/2}A_0^{-2}(R)]\right\}\Big/\left[8 + \frac{\epsilon\kappa^2(R)}{Q'^2 A_0^3(R)}\right]. \quad (32)$$

Hence, inserting equation (32) into equation (31), we obtain the amplitude of the excitation (soliton)

$$|A - A_0|_m = \frac{\epsilon\kappa^2(R)}{2Q'^2 A_0^3(R)} + \frac{\epsilon Q'^{1/2}A_0^2(R)}{3\kappa(R)}$$
$$\times \frac{d}{dR}\left\{\ln\left[R^{1/2}A_0^{-2}(R)\right]\right\}$$
$$+ Z_m^2\left[4 + \frac{\epsilon\kappa^2(R)}{2Q'^2 A_0^3(R)}\right] \quad (33)$$

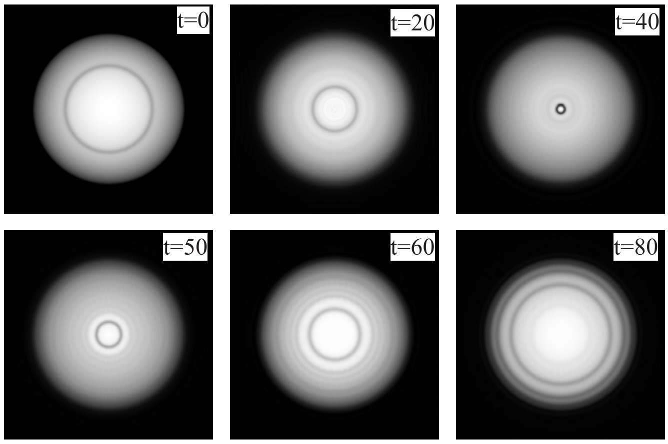

Fig. 1. Evolution of the ring dark soliton.

Equation (33) gives the variation of the amplitude of the ring dark soliton due to the radiation of sound caused by the ring curvature and the inhomogeneity of the BEC background.

To confirm our theoretical result, the evolution of ring dark soliton is also investigated numerically. For convenience, we rescale the wavefunction in equation (4) by $\Phi = \Phi'/\sqrt{Q'}$, then, still expresses the variable Φ' by Φ, equation (4) reduces to

$$i\frac{\partial\Phi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(r) + |\Phi|^2\right]\Phi. \quad (34)$$

Equation (34) is integrated numerically by means of the fourth-order Runge–Kutta scheme in time along with a second-order finite difference discretization in space. The spatial discretization step used in the simulations is typically $(\Delta x, \Delta y) = (0.1, 0.1)$. The time step of the integrator is $\Delta t = 0.0025$. The initial condition used to integrate equation (34) is given by equation (30) at $t = 0$ and with $Q' = 1$. We should note, however, that the initial condition given by equation (30) is not suitable for direct numerical integration due to the deviation from the background profile near the boundaries, where ϕ must be flat and A must approach the background solution A_0. Such perturbations result in non-desired pulses propagating inwards from the boundaries to the center of the interval. To avoid this problem, we correct the initial condition (30) in such a way that they approach the stationary background solution near boundaries (far from the initial soliton position $r = r_0$), and keep the shape of the soliton (30) in the vicinity of $r = r_0$. The parameters of $r_0 = 28.9$, $\epsilon = 0.1$, and $\Omega = 0.028$ are used. The evolution of the ring dark soliton is shown in Figure 1. It is clear that, as the time goes on, the ring soliton moves inward initially, attains the maximum contrast with minimum radius at $t \simeq 40$ and then bounces back. The ring soliton oscillates in the BEC cloud. When the ring oscillating in BEC, the ring soliton emit radiation because of the background inhomogeneity. The radiation is enhanced continuely when the ring is bounced back. Because the radiation loss, the ring will decay after about three oscillating. Those behaviour

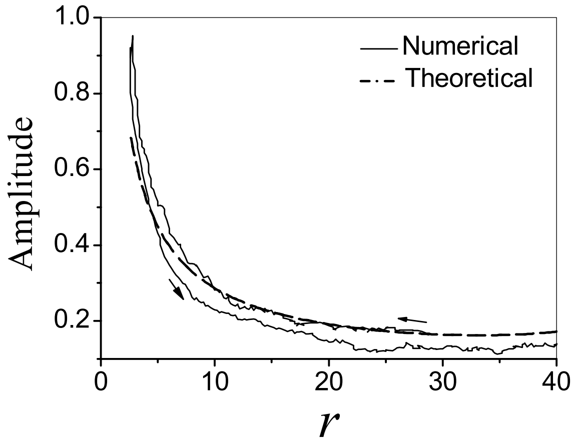

Fig. 2. Variation of soliton amplitude against radial r.

are approximately the same as predicted in reference [3]. Figure 2 shows the variation of the ring soliton amplitude $|A - A_0|_m$ against radius r. In Figure 2, both theoretical result given by equation (33) and the numerical result of equation (4) are show. We can find that, as the ring shrinks, the ring soliton amplitude increases. When the ring bounces back and expands outward, the ring soliton amplitude decreases. The overall variation law of the ring soliton amplitude given by our theoretical result is approximately in accordance with the numerical result. Strictly speaking, our theoretical results are suitable for the case of large ring radius (r) and far away from the turning points. This is clearly revealed in Figure 2: as the ring approaches the center (i.e., $r < 13$) and bounces back, the theoretical result deviates from the numerical result quantityly. The difference between the theoretical result and the numerical result increases as time goes on. However, Figure 2 indicates that our theoretical result gives a very good approximation for the ring soliton during the initial periods.

In summary, the evolution of ring dark soliton in a BEC with disc-shaped external potential is studied by both perturbation and numerical method. Theoretical analysis shows that the ring dark soliton is governed by a variable-coefficients KdV equation. The reduction to the KdV equation may be useful to understand the dynamics of ring dark soliton and will help to get a deeper insight into the physics of the dark soliton in inhomogeneous BEC. The analytical expressions for the evolution of soliton and emitted radiation profiles are also obtained explicitly. The numerical results confirm our theoretical results. All these results are not predicted in previous work [3].

This work is partly supported by the National Natural Science Foundation of China under grant Nos. 10475066 and 10347006, and by the Natural Science Foundation of Gansu province under grant No. 3ZS051-A25-013.

References

1. K.E. Strecker et al., Nature **417**, 150 (2002); L. Khaykovich et al., Science **296**, 1290 (2002)
2. S. Burger et al., Phys. Rev. Lett. **83**, 5198 (1999); J. Denschlag et al., Science **287**, 97 (2000); B.P. Anderson et al., Phys. Rev. Lett. **86**, 2926 (2001)
3. G. Theocharis et al., Phys. Rev. Lett. **90**, 120403 (2003)
4. P.O. Fedichev et al., Phys. Rev. A **60**, 3220 (1999)
5. D.L. Feder et al., Phys. Rev. A **62**, 053606 (2000)
6. Y.S. Kivshar, Xiaoping Yang, Phys. Rev. E **50**, R40 (1994)
7. T. Busch, J.R. Anglin, Phys. Rev. Lett. **83**, 2498 (1999)
8. G. Huang et al., Phys. Rev. A. **65**, 053605 (2002)
9. N.G. Parker et al., Phys. Rev. Lett. **90**, 220401 (2003)
10. D.J. Frantzeskakis et al., Phys. Rev. A **70**, 015601 (2004)
11. G. Huang, V.A. Makarov, M.G. Velarde, Phys. Rev. A **67**, 023604 (2003)
12. V.M. Perez-Garcia, H. Michinel, H. Herrero, Phys. Rev. A **57**, 3837 (1998)
13. T. Taniuti, Prog. Theor. Phys. Suppl. **55**, 1 (1974)
14. N. Asano, Prog. Theor. Phys. Suppl. **55**, 52 (1974)
15. R.S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves* (Cambridge University, Cambridge, U.K., 1997)
16. C.M. de Sterke, J.E. Sipe, *in Progress in Optics*, edited by E. Wolf (Elsevier Science B.V., Amsterdam, 1994), p. 203
17. V.I. Karpman, E.M. Maslov, Phys. Fluids **25**, 1686 (1982)
18. K. Ko, H.H. Kuehl, Phys. Fluids **23**, 834 (1980)